A local transform for trace monoids

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Abstract

We introduce a transformation for functions defined on the set of cliques of a trace monoid. We prove an inversion formula related to this transformation. It is applied in a probabilistic context in order to obtain a necessary normalization condition for the probabilistic parameters of invariant processes—a class of probabilistic processes introduced elsewhere, and intended to model an asynchronous and memoryless behavior.

1 Introduction

Trace monoids [3, 4, 6, 7], originally called commutation monoids in Combinatorics studies [2, 8], have received a great deal of interest from the theoretical Computer Science community because of their applications to the theory of concurrency. Recently, the author has introduced a probabilistic layer for trace monoids which motivates the present work.

1.1 Trace monoids

Let Σ be a finite a non empty alphabet, and let $I \subseteq \Sigma \times \Sigma$ be an irreflexive and symmetric binary relation, called *independence relation*. The monoid, that is to say, the semi-group with identity, generated by Σ and subject to the commutation relations $\alpha \cdot \beta = \beta \cdot \alpha$ for all $(\alpha, \beta) \in \mathcal{I}$ is called the *trace monoid* associated to the pair $(\Sigma, \mathcal{I})$. It is denoted $\mathcal{M}(\Sigma, \mathcal{I})$. Throughout the paper, we fix a trace monoid $\mathcal{M} = \mathcal{M}(\Sigma, \mathcal{I})$. If Σ^* denotes the free monoid of words on Σ, then $\mathcal{M}$ identifies with the quotient monoid $\Sigma^*/\sim$, where $\sim$ denotes the smallest congruence relation containing $\mathcal{I}$. The elements of $\mathcal{M}$ are called *traces*.

The left divisibility relation defined on $\mathcal{M}$ by:

$$\forall u, v \in \mathcal{M} \quad u \leq v \iff \exists r \in \mathcal{M} \quad v = u \cdot r\,,$$

is a partial ordering relation on $\mathcal{M}$.

An *independence clique*, or a *clique* for short, is a subset $c \subseteq \Sigma$ which all constitutive letters are mutually independent. In symbols:

$$\forall \alpha, \beta \in c \quad \alpha \neq \beta \Rightarrow (\alpha, \beta) \in \mathcal{I}.$$

Cliques correspond to the "parties commutatives" of [2]. Cliques are ordered by the inclusion of subsets: we write $c \leq c'$ and $c < c'$ for $c \subseteq c'$ and for $c \subsetneq c'$, respectively. The cardinality of a clique c is denoted $|c|$, and the set of cliques is denoted $\mathfrak{C}$.

The following definition is standard.

Definition 1.1. *Let c be a clique. An* enumeration *of c is a sequence $\varepsilon = (\alpha_1, \ldots, \alpha_r)$ of pairwise distinct letters $\alpha_i \in \Sigma$, and such that $c = \{\alpha_1, \ldots, \alpha_r\}$.*

We call ordered clique *any enumeration of some clique. The set of ordered cliques is denoted C.*

If $\varepsilon = (\alpha_1, \ldots, \alpha_r)$ is an ordered clique, we write $\lfloor \varepsilon \rfloor$ to denote the clique $\{\alpha_1, \ldots, \alpha_r\}$. Denoting by $|\varepsilon|$ the length of ε, and if $c = \lfloor \varepsilon \rfloor$, it is obvious that $|c| = |\varepsilon|$.

Let $\pi : \Sigma^* \to \mathcal{M}$ denote the canonical projection mapping, and let c be a clique. For any enumeration ε of c, the element $\pi(\varepsilon) \in \mathcal{M}$ is independent of ε. We will use the same notation c to denote the clique c, and the element $\pi(\varepsilon)$ for any enumeration ε of c.

1.2 Local transform

Let A be a commutative ring, and $\mathcal{A}$ be the set of functions $f : \mathfrak{C} \to A$. We will simply choose $A = \mathbb{R}$ for our probabilistic application below.

Definition 1.2. *For each $f \in \mathcal{A}$, the* local transform *of f is the element $g \in \mathcal{A}$ defined by:*

$$\forall c \in \mathfrak{C} \qquad g(c) = \sum_{c' \in \mathfrak{C} \,:\, c' \geq c} (-1)^{|c'| - |c|} f(c') \,. \tag{1}$$

The key result of the paper is an inversion formula for local transforms, stated in the following theorem.

Theorem 1.3. *Let $f \in \mathcal{A}$, and let g be the local transform of f. Then we have:*

$$\forall c \in \mathfrak{C} \qquad f(c) = \sum_{c' \in \mathfrak{C} \,:\, c' \geq c} g(c') \,. \tag{2}$$

1.3 Probabilistic application: a normalization condition

We shall apply the above inversion formula in a probabilistic context by considering some particular functions $f : \mathfrak{C} \to \mathbb{R}$. Let us quickly review the probabilistic elements that we are interested in.

The monoid $\mathcal{M}$ has a natural completion $\overline{\mathcal{M}}$ with respect to least upper bounds of chains. The added elements correspond to "infinite traces", which are the analogous of infinite words. There is a subset $\Omega \subseteq \overline{\mathcal{M}} \setminus \mathcal{M}$, the elements of which are called *samples*, and correspond to "fair" executions in a loose sense. The exact specification of Ω is irrelevant for our matter; we refer to [1] for details. We only need to know that Ω is equipped with a σ-algebra $\mathfrak{F}$ generated by the countable collection of *elementary cylinders* $\uparrow u$, for u ranging over $\mathcal{M}$, and defined by:

$$\forall u \in \mathcal{M} \qquad \uparrow u = \{\omega \in \Omega \;:\; \omega \geq u\}\,, \tag{3}$$

which are all non empty.

The author has introduced the notion of *invariant processes* in [1], as those probability measures $\mathbb{P}$ on $(\Omega, \mathfrak{F})$ with the following multiplicative property:

$$\forall u, v \in \mathcal{M} \qquad \mathbb{P}\big(\uparrow(u \cdot v)\big) = \mathbb{P}(\uparrow u) \cdot \mathbb{P}(\uparrow v)\,. \tag{4}$$

Invariant processes are a model of asynchronous and memoryless probabilistic processes. An invariant process $\mathbb{P}$ is entirely characterized by its finite collection of *characteristic numbers* $(p_\alpha)_{\alpha\in\Sigma}$, defined as follows:

$$\forall \alpha \in \Sigma \qquad p_\alpha = \mathbb{P}(\uparrow \alpha)\,. \tag{5}$$

An invariant process $\mathbb{P}$ is called *uniform* whenever all the characteristic numbers of $\mathbb{P}$ are equal. In this case, their common value is called *the* characteristic number of $\mathbb{P}$.

It was proposed in [1] a method for finding the adequate normalization conditions for a family $(p_\alpha)_{\alpha\in\Sigma}$ of real numbers to be indeed the family of characteristic numbers of an invariant process. The local transform introduced above provides an alternative method for obtaining a natural normalization condition. We strongly suspect that both methods actually yield equivalent normalization conditions, as verified on various examples, but this is not proved here.

The normalization condition that we introduce is related to the Möbius polynomial associated with the monoid $\mathcal{M}$, the definition of which is recalled now. Let $(T_\alpha)_{\alpha\in\Sigma}$ be a family of formal indeterminates. We denote by $\mathbb{Z}_{\mathcal{M}}\big[(T_\alpha)_{\alpha\in\Sigma}\big]$ the ring of formal polynomials with integer coefficients, generated by $(T_\alpha)_{\alpha\in\Sigma}$, and up to the commutation relations $T_\alpha \cdot T_\beta = T_\beta \cdot T_\alpha$ for all $(\alpha, \beta) \in \mathcal{I}$. The ring $\mathbb{Z}_{\mathcal{M}}\big[(T_\alpha)_{\alpha\in\Sigma}\big]$ is the quotient ring $\mathbb{Z}\big[(T_\alpha)_{\alpha\in\Sigma}\big]/\sim$, where $\sim$ is the smallest congruence containing all pairs $(T_\alpha T_\beta, T_\beta T_\alpha)$ for $(\alpha, \beta) \in \mathcal{I}$.

For each clique $c \in \mathfrak{C}$, the formal product $T_{\alpha_1} \cdot \ldots \cdot T_{\alpha_r} \in \mathbb{Z}_{\mathcal{M}}\big[(T_\alpha)_{\alpha\in\Sigma}\big]$, where $(\alpha_1, \ldots, \alpha_r)$ is any enumeration of c, is independent of the chosen enumeration. It is thus meaningful to put:

$$[c] = T_{\alpha_1} \cdot \ldots \cdot T_{\alpha_r}\,,$$

which is an element of $\mathbb{Z}_{\mathcal{M}}\big[(T_\alpha)_{\alpha\in\Sigma}\big]$, for any such enumeration of c. The *multivariate Möbius polynomial* [2] associated with $\mathcal{M}$ is the polynomial $\mu_{\mathcal{M}} \in \mathbb{Z}_{\mathcal{M}}\big[(T_\alpha)_{\alpha\in\Sigma}\big]$ defined by:

$$\mu_{\mathcal{M}} = \sum_{c\in\mathfrak{C}} (-1)^{|c|}[c]\,. \tag{6}$$

Note that the *evaluation* $P\big((t_\alpha)_{\alpha\in\Sigma}\big)$ of some polynomial $P\in\mathbb{Z}_{\mathcal{M}}\big[(T_\alpha)_{\alpha\in\Sigma}\big]$ on some family $(t_\alpha)_{\alpha\in\Sigma}$, consisting in substituting the values $(t_\alpha)_{\alpha\in\Sigma}$ to the family $(T_\alpha)_{\alpha\in\Sigma}$, is well defined provided the commutation relations $t_\alpha\cdot t_\beta = t_\beta\cdot t_\alpha$ for all $(\alpha,\beta)\in\mathcal{I}$ are satisfied. This is in particular the case if the family $(t_\alpha)_{\alpha\in\Sigma}$ takes values in a commutative ring.

As a particular case, the *monovariate Möbius polynomial* $\nu_{\mathcal{M}}\in\mathbb{Z}[X]$ is the standard polynomial obtained by substituting X to all indeterminates T_α in $\mu_{\mathcal{M}}$:

$$\nu_{\mathcal{M}}=\sum_{c\in\mathfrak{C}}(-1)^{|c|}X^{|c|}\,. \tag{7}$$

We will say that a complex number z is a *Möbius root of* $\mathcal{M}$ is z is a root of $\nu_{\mathcal{M}}$.

The relationship between characteristic numbers of invariant processes and the Möbius polynomials is stated in the following result, which provides the announced normalization condition.

Theorem 1.4. *Let $\mathbb{P}$ be an invariant process associated to $\mathcal{M}$, and let $(p_\alpha)_{\alpha\in\Sigma}$ be the collection of characteristic numbers of $\mathbb{P}$. Then:*

$$\mu_{\mathcal{M}}\big((p_\alpha)_{\alpha\in\Sigma}\big)=0\,. \tag{8}$$

In particular, if $\mathbb{P}$ is invariant uniform of characteristic number p, then p is a Möbius root of $\mathcal{M}$.

Note that we do not claim, say for the uniform case, that any Möbius root $p\in[0,1]$ of $\nu_{\mathcal{M}}$ is the characteristic number of some invariant and uniform process. Actually, it is not the case in general. We give an example in § 4 of a trace monoid with two Möbius roots in the open interval $(0,1)$, but for which we can prove that there is a unique invariant and uniform process.

It is a conjecture, first formulated by J. Mairesse, that p is the characteristic number of an invariant and uniform process if and only if p is the Möbius root of smallest modulus—it is known that $\nu_{\mathcal{M}}$ has indeed a unique and positive real root of smallest modulus, as proved through different methods in [5, 6]. This conjecture implies in particular the uniqueness of invariant and uniform processes on a given monoid, a reasonable statement.

1.4 Organization of the paper

In the remaining of the paper, we first give the proofs of the two previous results: the proof of Theorem 1.3 is given in § 2, the proof of Theorem 1.4 is given in § 3. Finally, in § 4, we give an example showing that not any Möbius root in $[0,1]$ is the characteristic number of an invariant and uniform process.

2 Proof of the inversion theorem

If c and c' are two cliques such that $c\cap c'=\emptyset$ and $c\cup c'\in\mathfrak{C}$, and only in this case, we use the notation $c\cdot c'$ to denote: $c\cdot c'=c\cup c'$. Hence: $|c\cdot c'|=|c|+|c'|$.

For any clique c, we put:

$$\mathfrak{C}_c = \{c' \in \mathfrak{C} \;:\; c' \geq c\}\,, \qquad \mathfrak{D}_c = \{e \in \mathfrak{C} \;:\; e \cap c = \emptyset \wedge e \cup c \in \mathfrak{C}\}\,.$$

It is then obvious that we have:

$$\mathfrak{C}_c = \{c \cdot e \;:\; e \in \mathfrak{D}_c\}\,. \tag{9}$$

Let $f \in \mathcal{A}$, and let $g \in \mathcal{A}$ be the local transform of f as given in Definition 1.2. Let $c \in \mathfrak{C}$, and denote by q_c the following quantity:

$$q_c = \sum_{c' \in \mathfrak{C}_c} g(c')\,.$$

We have to prove that $q_c = f(c)$. We compute as follows, using (9) and (1):

$$q_c = \sum_{e \in \mathfrak{D}_c} g(c \cdot e) = \sum_{e \in \mathfrak{D}_c} \sum_{e' \in \mathfrak{D}_{c \cdot e}} (-1)^{|e'|} f(c \cdot e \cdot e')\,.$$

For each integer $k \geq 0$, let λ_k denote the contribution to the above double summation of all cliques of the form $c \cdot e \cdot e'$ and such that $|e \cdot e'| = k$, so that we have:

$$q_c = \sum_{k \geq 0} \lambda_k\,. \tag{10}$$

For $k = 0$, the only contribution is obtained for $e = e' = \emptyset$, and thus $\lambda_0 = f(c)$.

Let $k > 0$. We have:

$$\lambda_k = \sum_{j=0}^{k} \sum_{\substack{e \in \mathfrak{D}_c \\ |e| = j}} \sum_{\substack{e' \in \mathfrak{D}_{c \cdot e} \\ |e'| = k-j}} (-1)^{k-j} f(c \cdot e \cdot e')\,. \tag{11}$$

In order to handle the double summations, we relate the cliques that appear in the sums to their enumerations (Definition 1.1). Recall that C denotes the set of ordered cliques, and define:

$$\forall c \in \mathfrak{C} \qquad D_c = \{\varepsilon \in C \;:\; \lfloor \varepsilon \rfloor \in \mathfrak{D}_c\}\,.$$

It is obvious that we have, for any integer $j \geq 0$:

$$\#\big\{e \in \mathfrak{D}_c \;:\; |e| = j\big\} = \frac{1}{j!} \#\big\{\varepsilon \in D_c \;:\; |\varepsilon| = j\big\}\,. \tag{12}$$

Using (11)(12), we have thus:

$$\lambda_k = \sum_{j=0}^{k} \frac{(-1)^{k-j}}{(k-j)!\,j!} \sum_{\substack{\varepsilon \in D_c \\ |\varepsilon| = j}} \sum_{\substack{\varepsilon' \in D_{c \cdot \lfloor \varepsilon \rfloor} \\ |\varepsilon'| = k-j}} f\big(c \cdot \lfloor \varepsilon \rfloor \cdot \lfloor \varepsilon' \rfloor\big)\,. \tag{13}$$

For $j \in \{0, \ldots, k\}$, denote by K_j the quantity:

$$K_j = \sum_{\substack{\varepsilon \in D_c \\ |\varepsilon| = j}} \sum_{\substack{\varepsilon' \in D_{c \cdot \lfloor \varepsilon \rfloor} \\ |\varepsilon'| = k-j}} f\big(c \cdot \lfloor \varepsilon \rfloor \cdot \lfloor \varepsilon' \rfloor\big) \,.$$

For any $j \in \{0, \ldots, k\}$, the cliques $c \cdot \lfloor \varepsilon \rfloor \cdot \lfloor \varepsilon' \rfloor$, for $(\varepsilon, \varepsilon')$ ranging over the above prescribed sets describe $k!$ times the following set of cliques:

$$H_k(c) = \{c' \in \mathfrak{C}_c \;:\; |c'| = |c| + k\} \,.$$

Therefore K_j is independent of j, and given by:

$$K_j = k! \times K \,, \quad \text{with } K = \sum_{c' \in H_k(c)} f(c') \,. \tag{14}$$

Combining (13)(14) yields:

$$\lambda_k = (-1)^k K \sum_{j=0}^{k} (-1)^j \frac{k!}{(k-j)!\, j!} \,.$$

Since $k > 0$, the binomial formula yields $\lambda_k = 0$. Going back to (10), we obtain $q_c = f(c)$, which was to be proved. The proof of Theorem 1.3 is complete.

3 Proof of the normalization condition

In this section, referring to the notions introduced in § 1, we consider an invariant process $\mathbb{P}$ with $(p_\alpha)_{\alpha \in \Sigma}$ as characteristic numbers.

Let $f : \mathfrak{C} \to \mathbb{R}$ be the function defined by:

$$\forall c \in \mathfrak{C} \qquad f(c) = \prod_{\alpha \in c} p_\alpha \,. \tag{15}$$

Since $\mathbb{P}$ is assumed to be invariant, it follows from the multiplicativity property (4) that f is equivalently given by:

$$\forall c \in \mathfrak{C} \qquad f(c) = \mathbb{P}(\uparrow c) \,. \tag{16}$$

We recall that any finite trace $u \in \mathcal{M}$ has a unique *Cartier-Foata* decomposition, that is to say, any $u \in \mathcal{M}$ is associated with a unique sequence $(c_1, \ldots, c_r)$ of cliques, such that the following two properties are satisfied:

1. $u = c_1 \cdot \ldots \cdot c_r$;
2. For all $j \in \{2, \ldots, r\}$, the following holds:

$$\forall \alpha \in c_j \quad \exists \beta \in c_{j-1} \quad \neg\big((\alpha, \beta) \in \mathcal{I}\big) \,. \tag{17}$$

What is called today the Cartier-Foata decomposition was originally called the "V-décomposition" in [2]. It is convenient to consider that the Cartier-Foata decomposition of each trace u is actually infinite, by adding the empty clique infinitely many times after the last non empty clique. It is well known that Cartier-Foata decomposition extends to infinite traces. Indeed, if $w = \bigvee_{k\geq 0} u_k$, where $(u_k)_{k\geq 0}$ is an increasing sequence in $\mathcal{M}$, then for all integer $j \geq 0$, the j^{th} element in the decomposition of u_k is eventually constant when k grows, equal to some clique c_j. It is then seen that the sequence $(c_j)_{j\geq 0}$ thus obtained satisfies:

1. $w = \bigvee_{j\geq 0} c_1 \cdot \ldots \cdot c_j$;
2. For all $j \geq 2$, property (17) holds.

Let $\mathfrak{C}^*$ denote the set of *non empty* cliques. Considering the first clique $C(\omega)$ in the Cartier-Foata decomposition of a sample $\omega \in \Omega$ defines a mapping:

$$C : \Omega \to \mathfrak{C}^* .$$

The combinatorial results of [2, Chap. I] have the following consequence.

Lemma 3.1. *Let $c \in \mathfrak{C}^*$. Then we have:*

$$\{\omega \in \Omega \ : \ C(\omega) = c\} = \uparrow c \setminus \bigcup_{\alpha\in\Sigma \,:\, c\cdot\alpha\in\mathfrak{C}} \uparrow c' . \tag{18}$$

According to our conventions, it is understood in (18) *that the range of letter α is such that $\alpha \notin c$.*

In particular, Lemma 3.1 proves that C is a measurable mapping, when equipping the finite set $\mathfrak{C}^*$ with its discrete σ-algebra.

Proposition 3.2. *Let $f : \mathfrak{C} \to \mathbb{R}$ be the function defined in* (15)(16) *relatively to some invariant process $\mathbb{P}$, and let $g : \mathfrak{C} \to \mathbb{R}$ be the local transform of f. Then we have:*

$$\forall c \in \mathfrak{C}^* \qquad \mathbb{P}(C = c) = g(c) . \tag{19}$$

Proof. Fix $c \in \mathfrak{C}^*$. Let $(\alpha_j)_{1\leq j\leq k}$ denote a family of pairwise distinct letters such that:

$$\{\alpha \in \Sigma \ : \ \alpha \notin c \wedge c \cup \{\alpha\} \in \mathfrak{C}\} = \{\alpha_1, \ldots, \alpha_k\} .$$

For $j \in \{1, \ldots, k\}$, put $A_j = \uparrow(c \cdot \alpha_j)$. According to (18), we have then:

$$\mathbb{P}(C = c) = \mathbb{P}(\uparrow c) - \delta, \quad \text{with } \delta = \mathbb{P}\Big(\bigcup_{j=1}^{k} A_j\Big) . \tag{20}$$

Poincaré inclusion-exclusion principle yields:

$$\delta = \sum_{r=1}^{k} (-1)^{r+1} \sum_{1\leq j_1\leq \ldots \leq j_r\leq k} \mathbb{P}\big(A_{j_1} \cap \cdots \cap A_{j_r}\big) . \tag{21}$$

We observe that the intersection $A_{j_1} \cap \cdots \cap A_{j_r}$ is either empty if $c \cup \{\alpha_{j_1}, \ldots, \alpha_{j_r}\}$ is not a clique, and equal to $\uparrow \big(c \cdot \{\alpha_{j_1}, \ldots, \alpha_{j_r}\}\big)$ otherwise. Therefore:

$$\sum_{1 \leq j_1 \leq \ldots \leq j_r \leq k} \mathbb{P}\big(A_{j_1} \cap \cdots \cap A_{j_r}\big) = \sum_{c' \in \mathfrak{C} \,:\, c' \geq c \wedge |c'-c|=r} \mathbb{P}(\uparrow c')\,. \tag{22}$$

Combining (21)(22) yields:

$$\begin{aligned}\delta &= \sum_{r=1}^{k} \sum_{c' \in \mathfrak{C} \,:\, c' \geq c \wedge |c'-c|=r} (-1)^{|c'|-|c|+1} \mathbb{P}(\uparrow c') \\ &= \sum_{c' \in \mathfrak{C} \,:\, c' > c} (-1)^{|c'|-|c|+1} \mathbb{P}(\uparrow c')\,.\end{aligned}$$

Going back to (20), we get:

$$\mathbb{P}(C = c) = f(c) + \sum_{c' \in \mathfrak{C} \,:\, c' > c} (-1)^{|c'|-|c|} f(c') = g(c)\,.$$

This completes the proof of the proposition. □

We now prove Theorem 1.4. It is obvious that (8) implies the statement for the uniform case, hence we focus on (8) only. Since the random variable C takes its values in $\mathfrak{C}^*$, the total probability law yields:

$$\sum_{c \in \mathfrak{C}^*} \mathbb{P}(C = c) = 1.$$

According to Proposition 3.2, this can also be written:

$$\sum_{c \in \mathfrak{C}^*} g(c) = 1. \tag{23}$$

Thanks to the inversion formula, we have:

$$f(\emptyset) = \sum_{c \in \mathfrak{C}} g(c)\,. \tag{24}$$

But $f(\emptyset) = \mathbb{P}(\Omega) = 1$. Hence (23)(24) yield: $g(\emptyset) = 0$. Comparing Definition 1.2 and the definition (6) of μ_M immediately yields $g(\emptyset) = \mu_{\mathcal{M}}\big((p_\alpha)_{\alpha \in \Sigma}\big)$, whence the result (8). The proof of Theorem 1.4 is complete.

4 An example with two Möbius roots in $(0, 1)$ and yet a unique uniform process

In this section we provide an example showing that not any Möbius root, even in the open interval $(0, 1)$, is associated with some invariant and uniform process. If the conjecture stating that only the Möbius root of smallest modulus corresponds to the characteristic number of some invariant and uniform process is established, then our example becomes a simple particular case.

Let $\Sigma = \{\alpha_1, \ldots, \alpha_5\}$, where α_i are 5 distinct letters. Consider the commutation relations $\alpha_i \cdot \alpha_j = \alpha_j \cdot \alpha_i$ for all $i, j \in \{1, \ldots, 5\}$ such that $|i - j| \geq 2$. In the framework of *multi-sites systems* introduced in [1], the associated monoid $\mathcal{M}$ corresponds to the monoid of finite trajectories associated to the 5-sites system $(S^1, \ldots, S^5)$ given by:

$$S^1 = \{x_5, x_1\}\,, \qquad\qquad S^i = \{x_{i-1}, x_i\} \quad \text{for } i \in \{2, \ldots, 5\},$$

where $x_1, \ldots, x_5$ are five pairwise distinct symbols. The correspondence between the x_i and the α_i is given by:

$$\alpha_1 = \begin{pmatrix} x_1 \\ x_1 \\ \emptyset \\ \emptyset \\ \emptyset \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} \emptyset \\ x_2 \\ x_2 \\ \emptyset \\ \emptyset \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} \emptyset \\ \emptyset \\ x_3 \\ x_3 \\ \emptyset \end{pmatrix} \quad \alpha_4 = \begin{pmatrix} \emptyset \\ \emptyset \\ \emptyset \\ x_4 \\ x_4 \end{pmatrix} \quad \alpha_5 = \begin{pmatrix} x_5 \\ \emptyset \\ \emptyset \\ \emptyset \\ x_5 \end{pmatrix}.$$

Assume that $\mathbb{P}$ is an invariant and uniform process associated with $\mathcal{M}$, of characteristic number p. Applying the method described in [1], we obtain that the following quantity is finite:

$$p \sum_{k \geq 0} p^k \sum_{j_1, \ldots, j_k \geq 0} p^{j_1 + \cdots + j_k} \sum_{l_1, \ldots, l_{j_1 + \cdots + j_k} \geq 0} p^{l_1 + \cdots + l_{j_1 + \cdots + j_k}} < \infty\,. \tag{25}$$

Since $p > 0$, computing the chained geometric sums yields successively:

$$p < 1\,, \qquad\qquad \frac{p}{1-p} < 1\,, \qquad\qquad \frac{p}{1 - \frac{p}{1-p}} < 1\,,$$

which is finally equivalent, given that $p > 0$, to:

$$0 < p < \frac{3 - \sqrt{5}}{2}\,. \tag{26}$$

On the other hand, the Möbius polynomial of $\mathcal{M}$ is given by:

$$\nu_{\mathcal{M}} = 1 - 5X + 5X^2\,, \tag{27}$$

with 2 roots in $(0, 1)$, namely $\rho_1 = \frac{5 - \sqrt{5}}{10}$ and $\rho_2 = \frac{5 + \sqrt{5}}{10}$. But only ρ_1 satisfies (26), and therefore ρ_2 is not the characteristic number of any invariant and uniform process. We refer to [1] for the proof of the existence of an invariant and uniform process with ρ_1 as characteristic number.

5 Conclusion

We have established an inversion formula for the *local transform*, a transformation that operates on the set of functions $\mathfrak{C} \to A$, where $\mathfrak{C}$ is the set of cliques of a trace monoid, and A is a commutative ring. Based on this inversion formula, we have proved that the family of characteristic numbers of an invariant process defined on the trace monoid cancels the Möbius polynomial associated with the monoid. This is interpreted as a normalization condition for the characteristic numbers of invariant processes.

If a given family of real numbers cancel the Möbius polynomial, it is however not obvious to determine whether this family is indeed the family of characteristic numbers of some invariant process. We have given the example of a monoid over an alphabet of 5 letters where the monovariate Möbius polynomial has two distinct roots in the open interval $(0, 1)$, whereas there is only one invariant and uniform process—that is to say, with equal characteristic numbers. It is conjectured that, for any trace monoid, only the Möbius root of smallest modulus is the characteristic number of an invariant and uniform process.

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